NONLINEAR COMPRESSED SENSING WITH APPLICATION TO PHASE RETRIEVAL

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ABSTRACT

We extend the ideas of compressed sensing to nonlinear measurement systems. In particular, we treat the problem of minimizing a general continuously differentiable function subject to sparsity constraints. We derive several different optimality criteria which are based on the notions of stationarity and coordinate-wise optimality. These conditions are then used to derive three numerical algorithms aimed at finding points satisfying the resulting optimality criteria: the iterative hard thresholding method and the greedy and partial sparse-simplex methods. The theoretical convergence of these methods and their relations to the derived optimality conditions are studied. We then specialize our algorithms to the problem of phase retrieval and develop an efficient method for retrieving a signal from its magnitude only measurements.

1. INTRODUCTION

Sparsity has long been exploited in signal processing and computer science. Despite the great interest in exploiting sparsity in various applications, most of the work to date has focused on recovering a sparse vector $\mathbf{x}$ from linear measurements of the form $\mathbf{b} = \mathbf{A}\mathbf{x}$ (as in standard compressed sensing problems).

In this work we study the more general problem of minimizing a continuously differentiable objective function $f : \mathbb{R}^n \to \mathbb{R}$ subject to a sparsity constraint:

$$\text{(P):} \quad \min f(\mathbf{x}) \quad \text{s.t.} \quad \|\mathbf{x}\|_0 \leq s.$$

Here $\|\mathbf{x}\|_0$ is the ℓ_0 norm of $\mathbf{x}$, which counts the number of its nonzero components. We do not assume that f is convex. This, together with the fact that the constraint function is nonconvex and not continuous, renders the problem difficult. We study necessary optimality conditions for (P) and develop algorithms that find points satisfying these conditions.

More specifically, we derive 3 classes of necessary optimality conditions: basic feasibility, L-stationarity, and coordinate-wise (CW) optimality. We then show that CW-optimality implies L-stationarity for suitable values of L, and they both imply basic feasibility. We also present two classes of algorithms for solving (P). The first is a generalization of iterative hard thresholding (IHT), and is based on the notion of L-stationarity. The second class of methods are based on the concept of CW-optimality. These are coordinate descent type algorithms which update the support at each iteration by one or two variables. Due to their resemblance with the celebrated simplex method for linear programming, we refer to these methods as "sparse-simplex" algorithms. As we show, these algorithms are as simple as IHT, while obtaining stronger optimality guarantees. Furthermore, they can be viewed as generalizations of matching pursuit techniques to the nonlinear setting.

Two examples of (P) that have been considered previously are compressed sensing and phase retrieval. The phase retrieval problem consists of recovering $\mathbf{x}$ from noisy measurements

$$y_i = |\langle \mathbf{a}_i, \mathbf{x} \rangle|^2 + w_i, \quad i = 1, \ldots, N \tag{1.1}$$

where w_i is noise, and $(\mathbf{a}_i)_{i=1}^N$ is a set of known vectors. Since only the magnitude of $\langle \mathbf{a}_i, \mathbf{x} \rangle$ is measured this problem is referred to as *phase retrieval*. Such problems arise in many areas of optics, where the detector can only measure the magnitude of the received optical wave. Several important applications of phase retrieval include X-ray crystallography, transmission electron microscopy and coherent diffractive imaging [8, 4, 3].

Many methods have been developed for phase recovery [4] which often rely on prior information on the signal, such as positivity or support constraints. One of the most popular techniques is based on alternating projections, where the signal estimate is transformed back and forth between the object and the Fourier domains. The prior information and observations are used in each domain to form the next estimate. Two of the main approaches of this type are Gerchberg-Saxton and Fienup [2]. In general, these methods are not guaranteed to converge, and often require careful parameter selection.

To circumvent the difficulties associated with alternating projections, more recently, phase retrieval problems have been treated by assuming sparsity on the input [6] and using methods based on semidefinite programming (SDP) [1, 9, 5, 7]. However, due to the increase in dimension created by the matrix lifting procedure, the SDP approach is not suitable for large-scale problems. Furthermore, it has no general optimality guarantees and often does not work satisfactory.

Our results for the general nonlinear recovery problem can be specialized to phase retrieval leading to an efficient method which also yields good recovery performance. We refer to our algorithm as GESPAR: GrEedy Sparse PhAse Retrieval. We demonstrate through numerical simulations that the proposed algorithm is both efficient and more accurate than current techniques. We will also show results of using GESPAR for solving the phase retrieval problem associated with a variety of different problems in optics including sub-wavelength coherent diffractive imaging, Ankylography, and recovery of optical signals in coupled waveguide arrays.

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2. REFERENCES

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